Computational Complexity of the Continuous Wavelet Transform in TwoDimensions

Romain Murenzi,[1] Lance Kaplan,[1,2] Jean-Pierre Antoine,[3] and Fernando Mujica[1,4]

murenzi@hubble.cau.edu, lmkaplan@hubble.cau.edu, antoine@fyma.ucl.ac.be, fmujica@eedsp.gatech.edu

Center for Theoretical Studies of Physical Systems[1,2]
Department of Physics[1] *and Engineering,*[2] *Clark Atlanta University*
Atlanta, GA 30314
Phone: (404) 880-8655

Institut de Physique Théorique[3]
Université Catholique de Louvain
B-1348 Louvain-la-Neuve, Belgium
Phone: +32 (10) 47 32 83

Center for Signal and Image Processing[4]
School of Electrical Engineering, Georgia Institute of Technology
Atlanta, GA 30332
Phone: (404) 894-2969

Abstract

The two-dimensional continuous wavelet transform (CWT) is characterized by a rotation parameter, in addition to the usual translations and dilations. The CWT has been interpreted as space-frequency representation of two-dimensional signals, where the translation corresponds to the position variable, and the inverse of the scale and the rotation, taken together, correspond to the spatial-frequency variable. The integral of the CWT's squared modulus, with respect to all variables, gives the energy of the original signal. Therefore, an integration on a subset of the parameters gives an energy density in the remaining variables. This paper deals with the implementation of the two basic densities, that is, the position (or aspect-angle) and scale-angle densities.

Introduction

The 2D continuous wavelet transform (CWT) has been used by a number of authors, in a wide variety of physical problems [1]. It has also been applied in designing CWT-based algorithms for detection and recognition of targets in a cluttered environment [2, 3, 4]. In all cases, its main use is in the *analysis* of images, that is, the detection of specific features. Examples of features are hierarchical structures or particular discontinuities (e.g., edges, filaments, contours, and boundaries between areas of different luminosity). Of course, the type of wavelet chosen depends on the precise application. In particular, the detection of directions requires the use of an oriented wavelet (Morlet), whereas an isotropic wavelet (Mexican Hat) suffices for pointwise analysis.

The 2D CWT is a representation of an image in a feature space with four parameters: scale, orientation, and position. From the energy-conservation theorem [5, 6], one defines various energy densities on any subset of the four variables.

We are currently investigating these energy densities in order to built a CWT 2D feature detector for a given target type in various conditions. Those features will then be used as input for a convolution neural network (CNN) algorithm [7]. These densities will also be used to design an ATR algorithm for detection, classification, and recognition of targets in FLIR and SAR imagery.

In the next section, we recall the definition of the CWT, and give its energy conservation property. We then discuss the implementation of the two basic energy densities, and the complexity of the related algorithm.

The 2D Continuous Wavelet Transform

The 2D CWT is obtained by projecting an image s onto a family of dilated, rotated, and translated versions of one function ψ, called the analyzing wavelet,

$$S(a,\theta,\vec{b}) = \frac{1}{\sqrt{c_\psi}}\langle \psi_{a,\theta,\vec{b}}|s\rangle \quad (1)$$
$$= \frac{1}{a\sqrt{c_\psi}}\int d^2\vec{x}\,\psi^*(r^{-\theta}(\frac{\vec{x}-\vec{b}}{a}))s(\vec{x}) \quad (2)$$
$$= \frac{a}{\sqrt{c_\psi}}\int d^2\vec{k}\,e^{i\vec{b}\cdot\vec{k}}\,\hat{\psi}^*(ar^{-\theta}(\vec{k}))\,\widehat{s}(\vec{k})\ ,(3)$$

where:

- The parameters a, θ, and $\vec{b}$ correspond to scale, rotation, and translation, respectively. The hat stands for the Fourier Transform, the * stands for the complex conjugation, and r^θ is the standard 2×2 rotation matrix.
- ψ verifies the so-called admissibility condition,

$$c_\psi = (2\pi)^2\int_{\mathbb{R}^2}\frac{d^2\vec{k}}{|\vec{k}|^2}|\widehat{\psi}(\vec{k})|^2 < \infty\ . \quad (4)$$

In practice, we will choose $c_\psi = 1$.

The wavelet transform satisfies the following energy conservation property,

$$\int_{\mathbb{R}^2} d^2\vec{x}\,|s(\vec{x})|^2 = \int_{\mathbb{R}^2} d^2\vec{k}\,|\hat{s}(\vec{k})|^2 \quad (5)$$
$$= \iiint \frac{da}{a^3}\,d\theta\,d^2\vec{b}\,|S(a,\theta,\vec{b})|^2\ .$$

This leads to the interpretation of $|S(a,\theta,\vec{b})|^2$ as an energy density of the signal s in position, scale, and orientation variables. There is a one-to-one map between the frequency space $\vec{k} = (k_x, k_y)$ and the scale-angle space (a,θ) by the identification $|\vec{k}| = a^{-1}$ and $\tan^{-1}(\frac{k_y}{k_x}) = \theta$. Therefore, $|S(a,\theta,\vec{b})|^2 = |\widetilde{S}(a^{-1},\theta,\vec{b})|^2$ is a space-frequency energy density. In other words, the CWT may be interpreted as a *phase space* representation of the signal.

It is clear that a partial integration of the CWT energy density over any subset of the variables gives an energy density in the remaining variables. Thus, there are four 1D densities, six 2D densities, and four 3D densities.

Among these, there are two basic densities: the position (or range-aspect) density and scale-angle density. In the following section, we discuss their implementation, and the complexity of the related computation.

Implementation of the Two Basic Densities

The first problem one faces when implementing the CWT is that of visualization. Indeed, $S(a,\theta,\vec{b})$ is a function of four variables: two position variables $\vec{b}\in\mathbb{R}^2$, and the pair $(a,\theta)\in\mathbb{R}^+_*\times[0,2\pi)\simeq\mathbb{R}^2_*$. One may say that the CWT has unfolded the signal from two to four dimensions; this feature explains its efficiency in decoupling singularities, but at the same time increases the memory requirements. As a consequence, some of the variables must be fixed for visualization of the CWT. There are many possibilities, but the interpretation of the parameter space as phase space given above suggests two natural ways of presenting the CWT, using two-dimensional sections of the parameter space:

(i) the *position representation*, where a and θ are fixed and the CWT is considered as a function of position $\vec{b}$ alone. The corresponding energy density is

$$E_{34}(b_x,b_y) = \int_0^\infty\int_0^{2\pi}\frac{da}{a^3}\,d\theta\,|S(a,\theta,b_x,b_y)|^2\ , \quad (6)$$

(ii) the *scale-angle representation*: for fixed $\vec{b}$, the CWT is considered as a function of scale and angle (a,θ), i.e. of spatial frequency

$$E_{12}(a,\theta) = \int_{\mathbb{R}^2} d^2\vec{b}\,|S(a,\theta,b_x,b_y)|^2\ . \quad (7)$$

The position representation is the standard one, and it is useful for the general purposes of image processing: detection of position, shape and contours of objects; pattern recognition; image filtering by resynthesis after elimination of unwanted features (for instance, noise). The scale-angle representation will be particularly interesting whenever scaling behavior (as in fractals) or angular selection is important, in particular when directional wavelets are used. In fact, both representations are needed for a full understanding of the properties of the CWT in all four variables.

Position (or Range-Aspect) Energy Density

The computation of $E_{34}(b_x,b_y)$ requires the calculation of $S(a_j,\theta_j,b_x,b_y)$ for $j = 1,....,N$, that is, N two-dimensional FFTs. But how do we choose the scale a_j and the angle θ_j? To answer this question one needs to define an appropriate sampling in the scale-angle variables.

Suppose ψ is a directional wavelet, that is, the effective support of its FT $\widehat{\psi}$ in spatial frequency space is contained in a convex cone of opening angle $\Delta\varphi$, with apex at the origin. We then say that $\widehat{\psi}$ is centered at $\vec{k}_o$ if

$$\vec{k}_o = \int d^2\vec{k}\, \vec{k} |\widehat{\psi}(\vec{k})|^2. \quad (8)$$

Its width in the x and y directions are given by $2w_x$, $2w_y$, where:

$$\begin{aligned} w_n &= \frac{1}{||\widehat{\psi}||} \left[\int d^2\vec{k}\, (k_n - k_{on})^2 |\widehat{\psi}(\vec{k})|^2 \right]^{1/2} \quad (9) \\ (n &= x, y)\,. \end{aligned}$$

Taking $\vec{k}_o = (0, k_o)$ along the y axis (see figure 1), we define the support of $\widehat{\psi}$ as the ellipse

$$\left(\frac{k_x}{w_x}\right)^2 + \left(\frac{k_y - k_o}{w_y}\right)^2 = 1. \quad (10)$$

Taking intercepts with the k_y-axis, we define the scale resolving power of ψ as

$$SRP(\psi) = \frac{k_o + w_y}{k_o - w_y}. \quad (11)$$

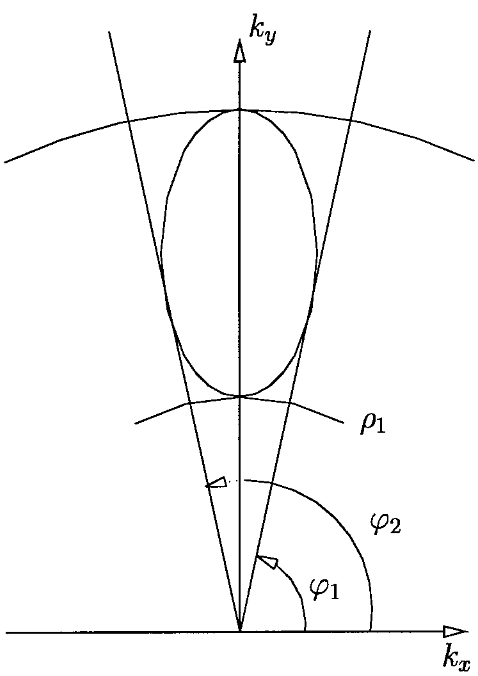

Figure 1: Directional wavelet in the frequency domain.

As for the angular resolving power $ARP(\psi) = \Delta\varphi$, we consider the tangents to that ellipse and obtain

$$\begin{aligned} ARP(\psi) &= 2\cot^{-1} \frac{\sqrt{k_o^2 - w_y^2}}{w_x} \quad (12) \\ &\simeq 2\cot^{-1} \frac{k_o}{w_x}, \text{ for } k_o \gg w_y. \quad (13) \end{aligned}$$

For a Morlet wavelet of anisotropy ϵ, this gives [5, 6]:

$$\begin{aligned} SRP(\psi_M) &= \frac{k_o\sqrt{2}+1}{k_o\sqrt{2}-1}\,, \quad (14) \\ ARP(\psi_M) &= 2\cot^{-1}(k_o\sqrt{\epsilon})\,. \quad (15) \end{aligned}$$

Using the angular- and scale-resolving powers, one obtains a tiling of the frequency space by the wavelet ψ, as indicated in figure 2.

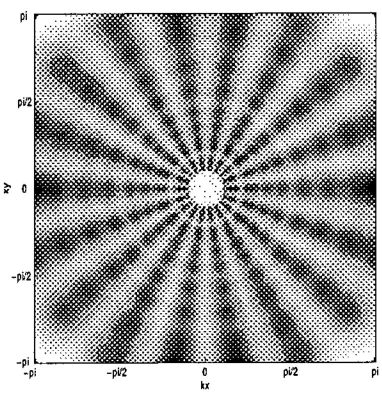

Figure 2: Tiling of frequency space using the parameters a and θ.

Suppose a signal s ranges in spatial frequency between ρ_m and ρ_M, and we want to analyze it with a wavelet ψ with scale support $[\rho_1, \rho_2]$. The extreme scales that are needed are, respectively, $a_m = \rho_m/\rho_1$ and $a_M = \rho_M/\rho_2$, and the scale range will be:

$$\frac{a_M}{a_m} = \frac{\rho_M/\rho_m}{\rho_2/\rho_1} = \frac{\Delta\rho_{\text{signal}}}{\Delta\rho_{\text{wavelet}}}\,. \quad (16)$$

Therefore, performing the discretization on a dyadic scale, the signal s will be completely analyzed by the family of wavelets ('filter bank') $\{\widehat{\psi}_{a_j,\theta_\ell}(\vec{k})\}$, where

$$\begin{aligned} a_j &= \frac{\rho_m}{\rho_1} \cdot 2^j, \quad j = 0, 1, \ldots, P-1, \quad (17) \\ \theta_\ell &= \Delta\varphi \cdot \ell, \quad \ell = 0, 1, \ldots, Q-1, \quad (18) \end{aligned}$$

with

$$\begin{aligned} P &= \text{integer part of } \log_2 \left(\frac{\Delta\rho_{\text{signal}}}{\Delta\rho_{\text{wavelet}}}\right) \quad (19) \\ Q &= \text{integer part of } \frac{2\pi}{\Delta\varphi}\,. \quad (20) \end{aligned}$$

Finally, the spatial energy density is

$$E_{34}(\vec{b}) = \sum_{j=0}^{P-1} \sum_{l=0}^{Q-1} |S(a_j, \theta_l, b_x, b_y)|^2\,. \quad (21)$$

Scale-Angle Density

The computation of $E_{12}(a,\theta)$ requires the calculation of $S(a,\theta,\vec{b}_j)$ for $j = 1, ..., N$. As the geometry in (b_x, b_y) plane is Cartesian, the sampling will be uniform, that is,

$$b_{x_j} = j\Delta x \ , \ b_{y_j} = j\Delta y, \tag{22}$$

where Δx and Δy are the horizontal and vertical bandwidths, respectively, of the wavelet in the space domain. We are still faced with the problem of calculating $S(a,\theta,\vec{b}_j)$ at a particular point. Let us consider the wavelet transform at $\vec{b} = \vec{b}_j$

$$S(a,\theta,\vec{b}_j) = \frac{1}{a}\int d^2\vec{x}\, \psi^*(r_{-\theta}(\frac{\vec{x}}{a}))s_{\vec{b}_j}(\vec{x}) \ , \tag{23}$$

where $s_{\vec{b}_j}(\vec{x}) \equiv s(\vec{x} + \vec{b}_j)$. In polar coordinates, we get

$$S(a,\theta,\vec{b}_j) = \int_0^\infty \int_0^{2\pi} d\rho d\varphi\, \frac{\rho}{a}\psi^*(\frac{\rho}{a}, \varphi - \theta)s_{\vec{b}_j}(\rho,\varphi) \ . \tag{24}$$

Performing the change of variables

$$\rho = e^u \text{ and } a = e^v, \tag{25}$$

we obtain

$$S(e^v,\theta,\vec{b}_j) = \int_{-\infty}^{+\infty}\int_0^{2\pi} du\, d\varphi\, e^{u-v}\psi^*(e^{u-v}, \varphi - \theta)\, e^u\, s_{\vec{b}_j}(e^u,\varphi) \ . \tag{26}$$

Denoting

$$G(u,\varphi) = \psi(e^u,\varphi), \ F(u,\varphi;\vec{b}_j) = e^u s_{\vec{b}_j}(e^u,\varphi), \tag{27}$$

Eq.(26) reduces to

$$S(e^v,\theta,\vec{b}_j) = \int_{-\infty}^{+\infty}\int_0^{2\pi} du\, d\varphi\, \overline{G}(u - v, \varphi - \theta)F(u,\varphi;\vec{b}_j) \ . \tag{28}$$

In this way, we have reduced the problem of computing $S(a,\theta,\vec{b}_j)$ into that of computing a convolution. The related complexity is N FFTs.

Conclusion

The calculation of the energy densities $E_{12}(a,\theta)$ and $E_{34}(\vec{b})$ uses a certain number N of 2D FFT. This number N depends on the support of the wavelet and signal in the space domain for $E_{12}(a,\theta)$, and the support of the wavelet and the signal in the spatial frequency domain for the density $E_{34}(\vec{b})$.

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Romain Murenzi, (404) 880-8655
Center for Theoretical Studies of Physical Systems, Department of Physics and Engineering
Atlanta, GA 30314

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